Symmetry transforms for ideal magnetohydrodynamics equilibria

Oleg I. Bogoyavlenskij
Department of Mathematics, Queen's University, Kingston, Canada K7L 3N6

(Received 11 October 2001; revised manuscript received 17 May 2002; published 22 November 2002)

A method for constructing ideal magnetohydrodynamics (MHD) equilibria is introduced. The method consists of the application of symmetry transforms to any known MHD equilibrium [O. I. Bogoyavlenskij, Phys. Rev. E. **62**, 8616, (2000)]. The transforms break the geometrical symmetries of the field-aligned solutions and produce continuous families of the nonsymmetric MHD equilibria. The method of symmetry transforms also allows to obtain MHD equilibria with current sheets and exact solutions with noncollinear vector fields **B** and **V**. A model of the nonsymmetric astrophysical jets outside of their accretion disks is developed. The total magnetic and kinetic energy of the jet is finite in any layer $c_1 < z < c_2$. The jet possesses current sheets and is highly collimated in view of a rapid decrease of the magnetic field **B** in the transverse direction. The method gives also the MHD equilibria that model ball lightning with dynamics of plasma inside the fireball.

DOI: 10.1103/PhysRevE.66.056410 PACS number(s): 52.30.Cv

I. INTRODUCTION

One of the possible approaches to the plasma confinement problem [2–4] and to the astrophysical jets problem [5–7] is to find and investigate the corresponding exact magnetohydrodynamics (MHD) equilibria. There are several known exact plasma equilibria [3,4]. However, until now there were no methods to obtain new solutions, especially equilibria that are bounded in the whole space. In the papers [8–11], the nonsymmetric plasma equilibria were derived that blow up as $x^2 + y^2 \to \infty$.

In this paper, we introduce a method for constructing ideal MHD equilibria which is based on the recently discovered [1] continuous symmetries of the MHD equilibrium equations. Applying these symmetries to any known equilibrium, we obtain continuous families of new MHD equilibria. For example, by applying the method to the exact solutions of the papers [12,13] we get the MHD equilibria that model ball lightning with dynamics of plasma inside the fireball.

The generic transforms [1] break the geometrical symmetries of the field-aligned plasma equilibria. For the magnetic analog of Hill's spherical vortex [14], the symmetry transforms break the axial symmetry of this equilibrium and give a continuous family of nonsymmetric MHD equilibria with toroidal magnetic surfaces and closed magnetic field lines. Applying the method to the well-behaved axially and helically symmetric plasma equilibria [1,15], we derive the nonsymmetric MHD equilibria which are bounded in the whole Euclidean space $\mathbb{R}^3$ and are rapidly decreasing as $x^2 + y^2 \to \infty$.

The method is used also to construct exact MHD equilibria with current sheets and contact (or tangential) discontinuities. These solutions are obtained by applying to the smooth MHD equilibria the symmetry transforms [1] that are discontinuous on some magnetic surfaces.

The method is different from the method of Backlund transforms for the soliton equations such as the Korteweg–de Vries (KdV) equation [16], the Kadomtsev-Petviashvili (KP) equation [17], the Sine-Gordon (SG) equation [18], etc. The method of Backlund transforms is based on the resolution of certain auxillary differential equations which usually cannot be solved explicitly. All soliton equations depend only on a part of spatial variables (one for KdV and SG and two for KP). Unlike the Backlund transforms, the method of symmetry transforms produces new solutions in explicit algebraic form and is applicable to the equations in all three spatial variables x, y, z.

We apply the method of symmetry transforms to develop a model of nonsymmetric astrophysical jets that are in the state of magnetohydrodynamics equilibrium. Such equilibria have to be global or have to satisfy the following necessary conditions in the cylindrical coordinates r, z, ϕ.

(a) The magnetic field **B**, the plasma velocity **V**, the pressure P, and density ρ are bounded in $\mathbb{R}^3$.

(b) The total kinetic and magnetic energy of plasma and its total mass are finite in any layer $c_1 < z < c_2$. The pressure $P \to p_0$ as $r \to \infty$.

(c) All magnetic field lines are bounded in the radial variable r.

The conditions (b) and (c) mean that the jet is localized in the x and y directions. The asymptotic value of pressure p_0 is the average pressure of the ambient medium. We suppose that gravitational effects can be neglected or in the case of constant density ρ the gravitational force $-\rho\,\mathrm{grad}\ \Psi$ is included in the pressure gradient.

We model the astrophysical jets outside of their accretion disks by the nonsymmetric global MHD equilibria that are bounded in the whole Euclidean space $\mathbb{R}^3$, satisfy the physical conditions (a), (b), (c), and have current sheets and tangential discontinuities. In Refs. [1], [15], I derived the exact global plasma equilibria which do not have current sheets and are axially and helically symmetric.

The system of magnetohydrodynamics equilibrium equations has the form

$$\rho(\mathbf{V}\cdot\ \mathrm{grad})\mathbf{V} + \frac{1}{\mu}\mathbf{B}\times \mathrm{curl}\mathbf{B} = -\,\mathrm{grad}\,P, \tag{1.1}$$

$$\mathrm{div}(\rho\mathbf{V}) = 0, \quad \mathrm{div}\,\mathbf{B} = 0, \quad \mathrm{curl}(\mathbf{V}\times\mathbf{B}) = \mathbf{0}, \tag{1.2}$$

where $\mathbf{B}$ is the magnetic vector field, μ is the constant magnetic permeability, $\mathbf{V}$ is the plasma velocity vector field, $\rho = \rho(\mathbf{x})$ is its density, and P is the pressure.

We consider both the incompressible and compressible plasma flows. The condition of incompressibility $\operatorname{div}\mathbf{V}=0$ is widely used in the MHD literature [19–22]. For example, it is applicable with a high accuracy for the subsonic plasma flows with Mach number $M \ll 1$, $M^2 = V^2/(\gamma P/\rho)$. Then the continuity equation $\operatorname{div}(\rho\mathbf{V})=0$ implies $\mathbf{V}\cdot \operatorname{grad}\rho(\mathbf{x})=0$. Hence the plasma density $\rho(\mathbf{x})$ is constant on the plasma streamlines.

For the compressible plasma flows, we suppose that plasma satisfies the ideal gas equation of state

$$P = C_0\rho^\gamma \exp(S/C_V), \quad \mathbf{V}\cdot \operatorname{grad} S = \mathbf{0}, \quad C_0 = P_0\rho_0^{-\gamma}, \tag{1.3}$$

where $S(\mathbf{x})$ is the density of entropy, $\gamma > 1$ is the adiabatic exponent, and C_V is the heat capacity at constant volume. We consider also the more general equations of state $P = C_0\rho^\gamma f(S)$.

II. THE SYMMETRY TRANSFORMS

For the noncollinear vector fields $\mathbf{B}$ and $\mathbf{V}$, the third equation (1.2) implies the existence of either magnetic surfaces or a magnetic foliation. Indeed, in any simply connected domain $E \subset \mathbb{R}^3$, $\mathbf{x}=(x,y,z)$, the equation $\operatorname{curl}(\mathbf{V}\times\mathbf{B})=\mathbf{0}$ yields

$$\mathbf{V}\times\mathbf{B} = \operatorname{grad}\psi(\mathbf{x}), \quad \psi(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} (\mathbf{V}\times\mathbf{B})\cdot d\mathbf{s}. \tag{2.1}$$

Hence the surfaces $\psi(\mathbf{x})=\text{const}$ are magnetic surfaces, because $\mathbf{B}\cdot \operatorname{grad}\psi=0$, $\mathbf{V}\cdot \operatorname{grad}\psi=0$ [19]. If the vector fields $\mathbf{B}$, $\mathbf{V}$ are defined only in some nonsimply connected domain D (for example, toroidal), then the function $\psi(\mathbf{x})$ (2.1) is multivalued in general. However, the differential form $d\psi$ is well defined. Indeed, we have

$$d\psi(\mathbf{Y}(\mathbf{x})) = [\mathbf{V}(\mathbf{x})\times\mathbf{B}(\mathbf{x})]\cdot\mathbf{Y}(\mathbf{x}) \tag{2.2}$$

for any tangent vector $\mathbf{Y}(\mathbf{x})$. The equation $d\psi(\mathbf{Y}(\mathbf{x}))=0$ defines an integrable foliation in the domain D which is generated by the vector fields $\mathbf{B}$ and $\mathbf{V}$.

For toroidal domains D (which are the most important for the tokamak applications) the function $\psi(\mathbf{x})$ is defined up to a constant nI, where n is an integer and I is the integral (2.1) over the shortest noncontractable loop of the torus.

For the incompressible flows, we suppose that the plasma density $\rho(\mathbf{x})$ is either constant or an arbitrary function of $\psi(\mathbf{x})$ (2.1), $\rho=\rho(\psi)$.

Any given smooth non-field-aligned MHD equilibrium $\mathbf{B}$, $\mathbf{V}$, ρ, P in $\mathbb{R}^3$ ($\mathbf{B}$ and $\mathbf{V}$ are noncollinear) defines a distribution of magnetic surfaces $\psi(\mathbf{x})=\text{const}$ in $\mathbb{R}^3$ [19]. Let E_m be the set of all incompressible equilibria that have the same magnetic surfaces as the given one. We introduce the transforms T: $E_m \to E_m$ that depend on arbitrary functions $a(\mathbf{x}) \neq 0$, $b(\mathbf{x})$, $c(\mathbf{x})$ that are constant on the magnetic surfaces $\psi(\mathbf{x})=\text{const}$. The $a(\mathbf{x})$, $b(\mathbf{x})$, $c(\mathbf{x})$, and $\rho(\mathbf{x})$ are either functions of $\psi(\mathbf{x})$ or, more generally, satisfy some functional equations $F(a(\mathbf{x}),\psi(\mathbf{x}))=0$, $G(b(\mathbf{x}),\psi(\mathbf{x}))=0$, $H(c(\mathbf{x}),\psi(\mathbf{x}))=0$, $R(\rho(\mathbf{x}),\psi(\mathbf{x}))=0$. For the MHD equilibria in a toroidal domain D (a tokamak), functions $a=a(\psi)$, $b=b(\psi)$, $c=c(\psi)$, and $\rho(\psi)$ are periodic with period I defined above.

The transforms T are defined by the formulas

$$\mathbf{B}_1 = b(\mathbf{x})\mathbf{B} + c(\mathbf{x})\sqrt{\mu\rho(\mathbf{x})}\,\mathbf{V},$$
$$\mathbf{V}_1 = \frac{c(\mathbf{x})}{a(\mathbf{x})\sqrt{\mu\rho(\mathbf{x})}}\mathbf{B} + \frac{b(\mathbf{x})}{a(\mathbf{x})}\mathbf{V}, \tag{2.3}$$
$$\rho_1(\mathbf{x}) = a^2(\mathbf{x})\rho(\mathbf{x}), \quad P_1 = CP + (C\mathbf{B}^2 - \mathbf{B}_1^2)/(2\mu),$$
$$b^2(\mathbf{x}) - c^2(\mathbf{x}) = C,$$

where $C \neq 0$ is a constant. Let us prove that transforms (2.3) define new solutions to the equilibrium equations (1.1) and (1.2). Let $h(\mathbf{x})$ be any function that is constant on the magnetic surfaces, for example, $a(\mathbf{x})$, $b(\mathbf{x})$, $c(\mathbf{x})$, or $\rho(\mathbf{x})$. Hence we have

$$\mathbf{B}\cdot \operatorname{grad} h(\mathbf{x})=0, \quad \mathbf{V}\cdot \operatorname{grad} h(\mathbf{x})=0. \tag{2.4}$$

Applying the classical identity $\mathbf{B}\times\operatorname{curl}\mathbf{B} = -(\mathbf{B}\cdot\operatorname{grad})\mathbf{B} + \operatorname{grad}(\mathbf{B}^2/2)$, we present Eq. (1.1) in the form

$$\rho(\mathbf{V}\cdot\operatorname{grad})\mathbf{V} - (\mathbf{B}\cdot\operatorname{grad})\mathbf{B}/\mu = -\operatorname{grad}(P + \mathbf{B}^2/(2\mu)). \tag{2.5}$$

Using formulas (2.3) and (2.4), we get

$$\rho_1(\mathbf{V}_1\cdot\operatorname{grad})\mathbf{V}_1 - (\mathbf{B}_1\cdot\operatorname{grad})\,\mathbf{B}_1/\mu + \operatorname{grad}(P_1 + \mathbf{B}_1^2/(2\mu))$$
$$= [b^2(\mathbf{x}) - c^2(\mathbf{x})][\rho(\mathbf{V}\cdot\operatorname{grad})\mathbf{V} - (\mathbf{B}\cdot\operatorname{grad})\mathbf{B}/\mu + \operatorname{grad}(P + \mathbf{B}^2/(2\mu))] = 0.$$

Thus the functions ρ_1, $\mathbf{B}_1$, $\mathbf{V}_1$, P_1 satisfy Eq. (2.5) and therefore Eq. (1.1).

Equations $\operatorname{div}(\rho_1\mathbf{V}_1)=0$ and $\operatorname{div}\mathbf{B}_1=0$ easily follow from Eq. (2.4) and $\operatorname{div}(\rho\mathbf{V})=0$, $\operatorname{div}\mathbf{B}=0$. Substituting Eq. (2.3), we obtain

$$\operatorname{curl}(\mathbf{V}_1\times\mathbf{B}_1) = \operatorname{curl}\left(\frac{C}{a(\mathbf{x})}\mathbf{V}\times\mathbf{B}\right) = \operatorname{grad}\frac{C}{a(\mathbf{x})}\times(\mathbf{V}\times\mathbf{B}) + \frac{C}{a(\mathbf{x})}\operatorname{curl}(\mathbf{V}\times\mathbf{B}).$$

Applying Eqs. (2.4), we find that vector field grad$(C/a(\mathbf{x}))$ is collinear with the vector field $\mathbf{V}\times\mathbf{B}$; hence grad$(C/a(\mathbf{x}))\times(\mathbf{V}\times\mathbf{B})=\mathbf{0}$. Therefore the equation curl$(\mathbf{V}\times\mathbf{B})=\mathbf{0}$ yields curl$(\mathbf{V}_1\times\mathbf{B}_1)=\mathbf{0}$. Hence formulas (2.3) for $C\neq 0$ define a new solution to Eqs. (1.1) and (1.2).

For $C\neq 0$, the transform (2.3) is invertible,

$$C\mathbf{B}=b(\mathbf{x})\mathbf{B}_1-c(\mathbf{x})\sqrt{\mu\rho_1(\mathbf{x})}\mathbf{V}_1,$$

$$C\mathbf{V}=\frac{-c(\mathbf{x})}{a_1(\mathbf{x})\sqrt{\mu\rho_1(\mathbf{x})}}\mathbf{B}_1+\frac{b(\mathbf{x})}{a_1(\mathbf{x})}\mathbf{V}_1, \tag{2.6}$$

where $a_1(\mathbf{x})=1/a(\mathbf{x})$. We shall refer to the transforms T (2.3) as the symmetries of the divergence-free MHD equilibrium equations (1.1) and (1.2).

For $C=0$, or $b(\mathbf{x})=\pm c(\mathbf{x})$, the transform (2.3) is not invertible and its range consists of the field-aligned solutions

$$\mathbf{B}_1=\pm\sqrt{\mu\rho_1(\mathbf{x})}\mathbf{V}_1,\quad P_1+\frac{1}{2\mu}\mathbf{B}_1^2=C_0=\text{const} \tag{2.7}$$

that are the known Chandrasekhar equipartition equilibria [20].

Remark 1. The new vector fields $\mathbf{B}_1$ and $\mathbf{V}_1$ (2.3) are linearly dependent on the original $\mathbf{B}$ and $\mathbf{V}$. Hence the new MHD equilibrium $\mathbf{B}_1$, $\mathbf{V}_1$, ρ_1, P_1 has the same magnetic surfaces as the original one $\mathbf{B}$, $\mathbf{V}$, ρ, P. This property shows that the symmetries (2.3) map the set E_m into itself. In Sec. IX, we apply the transforms (2.3) to obtain new MHD equilibria with noncollinear vector fields $\mathbf{B}$ and $\mathbf{V}$.

III. THE GROUPS OF SYMMETRIES

We consider the set G_m of all transforms (2.3) with $C\neq 0$ for which the smooth functions $a(\mathbf{x})$, $b(\mathbf{x})$, and $c(\mathbf{x})$ are constant on the magnetic surfaces for a given MHD equilibrium. Each transform (2.3) corresponds to a triple of functions (a,b,c) that satisfy the conditions

$$a(\mathbf{x})\neq 0,\quad b^2(\mathbf{x})-c^2(\mathbf{x})=\text{const}=C\neq 0. \tag{3.1}$$

The domain E_m for these transforms consists of all divergence-free MHD equilibria that have the same magnetic surfaces as the given one. Remark 1 and the invertibility of the transforms (2.3) for $C\neq 0$ prove that the range of these transforms is the same as their domain, E_m. Hence the composition of the transforms is well defined. Let us show that the composition assigns on the set G_m the structure of an Abelian group. Indeed, the composition of the transforms (2.3) is equivalent to the 3×3 matrix multiplication

$$\begin{pmatrix} a_2 & 0 & 0 \\ 0 & b_2 & c_2\sqrt{\mu\rho_1} \\ 0 & \dfrac{c_2}{\sqrt{\mu\rho_2}} & b_2\sqrt{\dfrac{\rho_1}{\rho_2}} \end{pmatrix}\times\begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_1 & c_1\sqrt{\mu\rho} \\ 0 & \dfrac{c_1}{\sqrt{\mu\rho_1}} & b_1\sqrt{\dfrac{\rho}{\rho_1}} \end{pmatrix}=\begin{pmatrix} a & 0 & 0 \\ 0 & b & c\sqrt{\mu\rho} \\ 0 & \dfrac{c}{\sqrt{\mu\rho_2}} & b\sqrt{\dfrac{\rho}{\rho_2}} \end{pmatrix},$$

where $a=a_2a_1$, $b=b_2b_1+c_2c_1$, $c=c_2b_1+b_2c_1$. Hence the multiplication of the triples has the form

$$(a_2,b_2,c_2)(a_1,b_1,c_1)=(a_2a_1,b_2b_1+c_2c_1,c_2b_1+b_2c_1), \tag{3.2}$$

which implies $C=b^2-c^2=C_2C_1\neq 0$. The unit triple is (1, 1, 0), the inverse transform (2.6) corresponds to the triple $(a,b,c)^{-1}=(a^{-1},C^{-1}b,-C^{-1}c)$. It is evident that the multiplication (3.2) is commutative and associative. Hence the transforms (2.3) form an Abelian group G_m.

The group G_m is infinite-dimensional and depends on the topology of the distribution of magnetic surfaces for the given MHD equilibrium because the functions $a(\mathbf{x})$, $b(\mathbf{x})$, $c(\mathbf{x})$ are constant on them. In Sec. IX, we show on a concrete example that, in general, the action of the group G_m on the set E_m is not transitive.

To study the structure of the group G_m, we introduce the parametrization $a(\mathbf{x})=\tau\exp\alpha(\mathbf{x})$, where $\alpha(\mathbf{x})$ is a smooth function that is constant on the magnetic surfaces and $\tau=\pm 1$. For $C=\sigma k^2$, $\sigma=\pm 1$, $k>0$, the second equation (3.1) is resolved in the form: $\sigma=1$, $b(\mathbf{x})=\eta k\,\mathrm{ch}\beta(\mathbf{x})$, $c(\mathbf{x})=\eta k\,\mathrm{sh}\beta(\mathbf{x})$; $\sigma=-1$, $b(\mathbf{x})=\eta k\,\mathrm{sh}\beta(\mathbf{x})$, $c(\mathbf{x})=\eta k\,\mathrm{ch}\beta(\mathbf{x})$, where $\eta=\pm 1$ and $\beta(\mathbf{x})$ is an arbitrary smooth function that is constant on the magnetic surfaces. Hence each transform (2.3) corresponds to a sextuple $[\alpha(\mathbf{x}),\beta(\mathbf{x}),k,\tau,\sigma,\eta]$. In view of the known identities for the hyperbolic functions ch$t$ and sh$t$, the multiplication (3.2) takes the simple form

$$[\alpha_1(\mathbf{x}),\beta_1(\mathbf{x}),k_1,\tau_1,\sigma_1,\eta_1][\alpha_2(\mathbf{x}),\beta_2(\mathbf{x}),k_2,\tau_2,\sigma_2,\eta_2]=[\alpha_1(\mathbf{x})+\alpha_2(\mathbf{x}),\beta_1(\mathbf{x})+\beta_2(\mathbf{x}),k_1k_2,\tau_1\tau_2,\sigma_1\sigma_2,\eta_1\eta_2].$$

Hence the group G_m is the direct sum

$$G_m = A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2 . \tag{3.3}$$

Here R^+ is the multiplicative group of positive numbers $k > 0$. The A_m is the additive Abelian group of smooth functions in $\mathbb{R}^3$ that are constant on the magnetic surfaces for the given MHD equilibrium. The group A_m is also a linear space and an associative algebra with respect to the multiplication of functions.

Remark 2. An additional algebraic structure. The group G_m has an additional structure that does not exist for the groups of symmetries of the soliton equations [16–18]. The G_m is a module over the associative algebra $A_m \oplus A_m$ with the multiplication defined by the multiplication of functions,

$$[\gamma(\mathbf{x}),\zeta(\mathbf{x})][\alpha(\mathbf{x}),\beta(\mathbf{x}),k,\tau,\sigma,\eta]$$
$$=[\gamma(\mathbf{x})\alpha(\mathbf{x}),\zeta(\mathbf{x})\beta(\mathbf{x}),k,\tau,\sigma,\eta]. \tag{3.4}$$

This operation results in the transform: $a(\mathbf{x}) \to \tau |a(\mathbf{x})|^{\gamma(\mathbf{x})}$, $b(\mathbf{x}) = \eta k \,\mathrm{ch}\beta(\mathbf{x}) \to \eta k \,\mathrm{ch}(\zeta(\mathbf{x})\beta(\mathbf{x}))$, $c(\mathbf{x}) = \eta k \,\mathrm{sh}\beta(\mathbf{x}) \to \eta k \,\mathrm{sh}(\zeta(\mathbf{x})\beta(\mathbf{x}))$.

Remark 3. Subgroups of the groups G_m. The intersection of all groups G_m for different MHD equilibria consists of the transforms (2.3) with constant coefficients a,b,c,

$$\mathbf{B}_1 = b\mathbf{B} + c\sqrt{\mu\rho(\mathbf{x})}\mathbf{V}, \quad \mathbf{V}_1 = \frac{c}{a\sqrt{\mu\rho(\mathbf{x})}}\mathbf{B} + \frac{b}{a}\mathbf{V}, \tag{3.5}$$

$$\rho_1(\mathbf{x}) = a^2\rho(\mathbf{x}), \quad P_1 = CP + (C\mathbf{B}^2 - \mathbf{B}_1^2)/(2\mu),$$

$$b^2 - c^2 = C \neq 0.$$

The transforms (3.5) are applicable to any MHD equilibria and form the three-dimensional subgroup

$$G_3 = \cap G_m = R \oplus R \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2 .$$

They correspond to the $\mathbf{x}$-independent sextuples $(\alpha,\beta,k,\tau,\sigma,\eta)$ where $a = \tau \exp \alpha$, $C = \sigma k^2 \neq 0$, and $b = \eta k \,\mathrm{ch}\beta$, $c = \eta k \,\mathrm{sh}\beta$ for $\sigma = 1$, and $b = \eta k \,\mathrm{sh}\beta$, $c = \eta k \,\mathrm{ch}\beta$ for $\sigma = -1$. Each group G_m (3.3) has exactly eight periodic elements that are defined by the sextuples (0, 0, 1, ± 1, ± 1, ± 1) and form the subgroup $\Gamma_8 = Z_2 \oplus Z_2 \oplus Z_2 \subset G_3 \subset G_m$; all periodic transforms T have order 2, $T^2 = \mathrm{id}$. The subgroup Γ_8 represents the eight components G_{mj}, $j = 1,...,8$, of the group G_m. The scale and reflection symmetries form the two-dimensional subgroup $\Gamma_2 = R \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2 \subset G_3 \subset G_m$ that consists of the sextuples $(\alpha,0,k,\tau,\sigma,\eta)$.

IV. THE SYMMETRY TRANSFORMS FOR THE FIELD-ALIGNED MHD EQUILIBRIA

For any field-aligned MHD equilibrium, we have

$$\mathbf{V} = \frac{\lambda(\mathbf{x})}{\sqrt{\mu\rho(\mathbf{x})}}\mathbf{B}, \tag{4.1}$$

where $\lambda(\mathbf{x})$ is some smooth function in $\mathbb{R}^3$. Let E_ℓ be the set of all divergence-free field-aligned MHD equilibria that have the same magnetic field lines as the given equilibrium (4.1). In this section, we introduce transforms T_ℓ: $E_\ell \to E_\ell$ that depend on the arbitrary functions $a_1(\mathbf{x}) \neq 0$, $b_1(\mathbf{x})$, $c_1(\mathbf{x})$, that are constant on the considered magnetic field lines. These functions are first integrals of the corresponding dynamical system

$$\dot{x} = B_x(x,y,z), \quad \dot{y} = B_y(x,y,z), \quad \dot{z} = B_z(x,y,z), \tag{4.2}$$

where B_x, B_y, B_z are the magnetic field $\mathbf{B}$ components. Such first integrals exist always if the trajectories of system (4.2) lie on some two-dimensional (magnetic) surfaces. In the paper [19], Moffatt proved that all smooth steady MHD equilibria possess magnetic surfaces. The only possible exceptions are the "force-free" equilibria

$$\mathbf{V} = c_1\mathbf{B}, \quad \rho = c_2, \quad \mathrm{curl}\mathbf{B} = c_3\mathbf{B}, \quad P + \rho\mathbf{V}^2/2 = c_4, \tag{4.3}$$

where c_1, c_2, c_3, c_4 are constants [19].

Equations (1.2) and $\mathrm{div}\,\mathbf{V} = 0$ imply $\mathbf{B}\cdot \mathrm{grad}\,\lambda(\mathbf{x}) = 0$ and $\mathbf{B}\cdot \mathrm{grad}\,\rho(\mathbf{x}) = 0$. Hence the functions $\lambda(\mathbf{x})$ and $\rho(\mathbf{x})$ are first integrals of the magnetic dynamical system (4.2).

The transforms T_ℓ are defined by the formulas

$$\rho_1(\mathbf{x}) = a_1^2(\mathbf{x})\rho(\mathbf{x}), \quad \mathbf{B}_1 = b_1(\mathbf{x})\mathbf{B}, \quad \mathbf{V}_1 = \frac{c_1(\mathbf{x})}{a_1(\mathbf{x})\sqrt{\mu\rho(\mathbf{x})}}\mathbf{B}, \tag{4.4}$$

$$P_1 = C_1 P + (C_1\mathbf{B}^2 - \mathbf{B}_1^2)/(2\mu), \quad \frac{b_1^2(\mathbf{x}) - c_1^2(\mathbf{x})}{1 - \lambda^2(\mathbf{x})} = C_1,$$

where $C_1 \neq 0$ is a constant. Let us prove that the $\mathbf{B}_1$, $\mathbf{V}_1$, ρ_1, P_1 (4.4) represent a new MHD equilibrium. For the field-aligned solutions (4.1), Eq. (1.1), Eq. (2.5) takes the form

$$\mu^{-1}(\lambda^2(\mathbf{x}) - 1)(\mathbf{B}\cdot \mathrm{grad})\mathbf{B} + \mathrm{grad}(P + \mathbf{B}^2/(2\mu)) = 0. \tag{4.5}$$

Applying transform (4.4) and using formula (4.5), we obtain

$$\rho_1(\mathbf{V}_1\cdot \mathrm{grad})\mathbf{V}_1 - (\mathbf{B}_1\cdot \mathrm{grad})\mathbf{B}_1/\mu + \mathrm{grad}(P_1 + \mathbf{B}_1^2/(2\mu))$$
$$= \mu^{-1}(c_1^2(\mathbf{x}) - b_1^2(\mathbf{x}))(\mathbf{B}\cdot \mathrm{grad})\mathbf{B}$$
$$+ C_1\,\mathrm{grad}(\mathbf{P} + \mathbf{B}^2/(2\mu)) = 0.$$

Hence functions ρ_1, $\mathbf{B}_1$, $\mathbf{V}_1$, and P_1 (4.4) satisfy Eq. (1.1). It is evident that functions (4.4) satisfy Eqs. (1.2). Hence transforms T_ℓ (4.4) map the set E_ℓ into itself. Transforms (4.4) are evidently invertible if $C_1 \neq 0$.

Transforms (4.4) are the restrictions of the transforms (2.3) onto the field-aligned equilibria. Formulas (4.4) imply $\mathbf{V}_1 = \lambda_1(\mathbf{x})\mathbf{B}_1/\sqrt{\mu\rho_1(\mathbf{x})}$ where $\lambda_1(\mathbf{x}) = c_1(\mathbf{x})/b_1(\mathbf{x})$. The last formula (4.4) yields $\lambda_1^2(\mathbf{x}) = 1 - C_1 b_1^{-2}(\mathbf{x})[1 - \lambda^2(\mathbf{x})]$. Hence for $C_1 \neq 0$ we have $|\lambda_1(\mathbf{x})| = 1$ if and only if $|\lambda(\mathbf{x})| = 1$. Therefore the Chandrasekhar solutions (2.7), $|\lambda(\mathbf{x})| = 1$, are invariant under transforms (4.4).

Remark 4. On the structure of the functions $a_1(\mathbf{x})$, $b_1(\mathbf{x})$, $c_1(\mathbf{x})$. These functions are first integrals of the magnetic dynamical system (4.2). The following can occur in different invariant domains $D \subset \mathbb{R}^3$ [2].

(1) The magnetic field lines (4.2) are dense on some closed magnetic surfaces defined by an equation $f(\mathbf{x}) = \text{const}$. Then these surfaces necessarily are tori $\mathbb{T}^2$ [2] and first integrals $a_1(\mathbf{x}) \neq 0$, $b_1(\mathbf{x})$, $c_1(\mathbf{x})$ are constant on them. Hence they are either smooth functions of f or, more generally, satisfy some equations $F(a_1(\mathbf{x}), f(\mathbf{x})) = 0$, $G(b_1(\mathbf{x}), f(\mathbf{x})) = 0$, $H(c_1(\mathbf{x}), f(\mathbf{x})) = 0$.

(2) All magnetic field lines either are closed curves or go to infinity—then the first integrals $a_1(\mathbf{x}) \neq 0$, $b_1(\mathbf{x})$, $c_1(\mathbf{x})$ are smooth functions of the two transversal variables. If first integrals $\lambda(\mathbf{x})$ and $\rho_1(\mathbf{x})$ are functionally independent, then the a_1, b_1, c_1 are smooth functions of them: $a_1 = a_1(\lambda, \rho)$, $b_1 = b_1(\lambda, \rho)$, $c_1 = c_1(\lambda, \rho)$.

(3) The magnetic field lines do not lie on any two-dimensional surfaces. This can happen only for the force-free equilibria (4.3) [19]. Then the functions $a_1(\mathbf{x})$, $b_1(\mathbf{x})$, $c_1(\mathbf{x})$ are constant.

Any field-aligned solution (4.1) for $\lambda(\mathbf{x}) \neq 1$ can be transformed by the symmetries (4.4) to a pure magnetic equilibrium, $\mathbf{B}_1 = b_1(\mathbf{x})\mathbf{B}$, $\mathbf{V}_1 = \mathbf{0}$, $c_1(\mathbf{x}) = 0$, $b_1^2(\mathbf{x}) = C_1(1 - \lambda^2(\mathbf{x}))$, and to a pure hydrodynamic equilibrium, $\mathbf{B}_1 = \mathbf{0}$, $\mathbf{V}_1 = c_1(\mathbf{x})\mathbf{B}/[a_1(\mathbf{x})\sqrt{\mu\rho(\mathbf{x})}]$, $b_1(\mathbf{x}) = 0$, $c_1^2(\mathbf{x}) = C_1(\lambda^2(\mathbf{x}) - 1)$.

Remark 5. The exact field-aligned MHD equilibria (4.4) with constant a_1, b_1, c_1 were first derived in Ref. [23] from the exact axially symmetric plasma equilibria found in Ref. [15]. Construction (4.4) with arbitrary smooth functions $a_1(\mathbf{x}) \neq 0$, $b_1(\mathbf{x}) \neq 0$ was first applied in Ref. [1] for the helically symmetric plasma equilibria.

V. BALL LIGHTNING MODEL WITH DYNAMICS OF PLASMA

In the papers [12,13], a model of ball lightning is developed where a steady plasma with velocity $\mathbf{V} = \mathbf{0}$ fills a spherical ball and the magnetic field $\mathbf{B}$ is axially symmetric inside the ball and vanishes outside of it. The model is based on an exact solution of the paper [12] which is given in terms of the spherical Bessel functions and the Legendre functions. In what follows, we generalize this model for the nonzero plasma velocity $\mathbf{V} \neq \mathbf{0}$.

In the cylindrical coordinates r, z, ϕ, the axially symmetric magnetic field $\mathbf{B}$ has the form [2]

$$\mathbf{B} = \frac{\psi_z}{r}\hat{\mathbf{e}}_r - \frac{\psi_r}{r}\hat{\mathbf{e}}_z + \frac{I}{r}\hat{\mathbf{e}}_\phi, \tag{5.1}$$

where $\psi(r,z)$ is a flux function, $\psi_z = \partial\psi/\partial z$, $\psi_r = \partial\psi/\partial r$, $I = I(r,z)$ describes the electric current density, and $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_z, \hat{\mathbf{e}}_\phi$ are the coordinate unit orts. For the axially symmetric solutions (5.1), the plasma equilibrium equations $\text{curl}\mathbf{B} \times \mathbf{B} = \mu\,\text{grad}\,P$, $\text{div}\,\mathbf{B} = 0$ are equivalent to the Grad-Shafranov equation $\psi_{rr} - \psi_r/r + \psi_{zz} + I(\psi)I'(\psi) + \mu r^2 p'(\psi) = 0$, where $I = I(\psi)$ and $P = p(\psi)$ are arbitrary functions of ψ and prime means the derivative with respect to ψ. The authors of Refs. [12], [13] assume $I(\psi) = \alpha\psi$, $p(\psi) = p_0 - c_0\psi$ and construct a solution $\psi(r,z)$ that satisfies the overdetermined boundary conditions $\psi|_{\partial V} = 0$, $\text{grad}\,\psi|_{\partial V} = 0$ on a spherical boundary $\partial V: R = a$, $R = \sqrt{r^2 + z^2}$. Hence $\mathbf{B}|_{\partial V} = \mathbf{0}$, $P|_{\partial V} = p_0$ and the solution is continued in the outer space $R > a$ by the trivial solution $\mathbf{B}(\mathbf{x}) = \mathbf{0}$, $P(\mathbf{x}) = p_0$ [12,13]. The exact solution of Ref. [12] can be represented in the form

$$\psi(r,z) = mr^2\left[1 + \frac{3 - x_1^2}{\cos x_1}\frac{1}{\alpha^2 R^2}\left(\cos(\alpha R) - \frac{\sin(\alpha R)}{\alpha R}\right)\right], \tag{5.2}$$

where $m = c_0\mu/\alpha^2$, $(3 - x_1^2)/\cos x_1 \approx -34.8145$, and $0 \leqslant R \leqslant a = x_1/\alpha$. Here $x_1 \approx 5.763\,459$ is the smallest positive root of the equation $\tan x = 3x/(3 - x^2)$ where $x = \alpha R$. Inside the ball $R \leqslant a$, the generic magnetic surfaces $\psi(r,z) = \text{const}$ are toroidal. The singular magnetic surfaces are the segment $r = 0$, $-a \leqslant z \leqslant a$ and the magnetic axis $r = r_1$, $z = 0$, $0 \leqslant \phi \leqslant 2\pi$, which is defined by the conditions $\psi_r = 0$, $\psi_z = 0$.

Let $\beta(\psi)$ be an arbitrary smooth function of ψ, $b_1 = \text{ch}\beta(\psi)$, $c_1 = \text{sh}\beta(\psi)$, $C_1 = b_1^2 - c_1^2 = 1$, and $a_1(\psi) \neq 0$ be another smooth function. Applying the corresponding symmetry transform (4.4) to the equilibrium (5.1), (5.2) we get new field-aligned sub-Alfvenic MHD equilibria

$$\mathbf{B}_1 = \text{ch}\beta(\psi)\mathbf{B}, \quad \mathbf{V}_1 = \frac{\text{sh}\beta(\psi)}{a_1(\psi)\sqrt{\mu}}\mathbf{B},$$

$$P_1 = p_0 - c_0\psi - \frac{1}{2\mu}\text{sh}^2\beta(\psi)\mathbf{B}^2, \tag{5.3}$$

with the plasma density $\rho_1(\mathbf{x}) = a_1^2(\psi(r,z))$. The MHD equilibria (5.3) are defined inside the ball $R \leqslant a$ and satisfy the boundary conditions $\mathbf{B}_1|_{\partial V} = \mathbf{0}$, $\mathbf{V}_1|_{\partial V} = \mathbf{0}$, $P_1|_{\partial V} = p_0$. Hence the equilibria are continued in the outer space $R > a$ by the trivial solution $\mathbf{B}(\mathbf{x}) = \mathbf{0}$, $\mathbf{V}(\mathbf{x}) = \mathbf{0}$, $P(\mathbf{x}) = p_0 = \text{const}$, $\rho_1(\mathbf{x}) = a_1^2(0) = \text{const}$. The field-aligned equilibria (5.3) model ball lightning with the variable plasma density and sub-Alfvenic dynamics of plasma inside the fireball. These equilibria manifest also the neutral modes for the stability analysis [13] of the ball lightning model [12,13]. Arguments of the papers [24,25] concerning stability of the sub-Alfvenic MHD equilibria are completely applicable to the solutions (5.3).

VI. BREAKING OF THE GEOMETRICAL SYMMETRIES

The symmetry transforms (4.4) have important applications connected with the breaking of the geometrical symmetries of the field-aligned MHD equilibria. Suppose that a field-aligned MHD equilibrium (4.1) possesses some geometrical symmetry—translational, axial, or helical—and that all magnetic field lines in an invariant domain D either are closed curves or go to infinity. Then the symmetries (4.4) are defined by first integrals $a_1(\mathbf{x})$, $b_1(\mathbf{x})$, $c_1(\mathbf{x})$ of the dynamical system (4.2), which depend on the two transversal variables. These functions generically are not invariant with respect to the above geometrical symmetries. Hence the new equilibrium $\rho_1, \mathbf{B}_1, \mathbf{V}_1, P_1$ (4.4) is nonsymmetric. This means that the symmetry transforms (2.3), (4.4) break the geometrical symmetry of the original field-aligned equilibrium ρ, $\mathbf{B}$, $\mathbf{V}$, P (4.1). In Sec. VIII, we present the global nonsymmetric MHD equilibria obtained by breaking of the helical symmetry.

In this section, we give an example of the axial symmetry breaking for a plasma equilibrium with toroidal magnetic surfaces and closed magnetic field lines. We consider the magnetic analog [26] of Hill's spherical vortex [14] for which the magnetic field $\mathbf{B}$ is axially symmetric and pure poloidal,

$$\mathbf{B}=\frac{\psi_z}{r}\hat{\mathbf{e}}_r-\frac{\psi_r}{r}\hat{\mathbf{e}}_z. \tag{6.1}$$

The notations are the same as in Eq. (5.1), $I\equiv 0$. Inside the ball $R\leqslant a$, $R=\sqrt{r^2+z^2}$, Hill's solution has the flux function

$$\psi(r,z)=c_0\mu r^2(R^2-a^2)/10, \quad P(\psi)=p_0-c_0\psi. \tag{6.2}$$

Outside of the ball, $R>a$, the flux function is

$$\hat{\psi}(r,z)=Ar^2(R^{-3}-a^{-3}), \quad A=-a^5c_0\mu/15, \quad p(\psi)=p_0. \tag{6.3}$$

The formulas (6.1)–(6.3) imply $\mathbf{B}(\mathbf{x})=\hat{\mathbf{B}}(\mathbf{x})$ for $|\mathbf{x}|=a$. In the outer space $R>a$, the magnetic field $\hat{\mathbf{B}}$ (6.1), (6.3) is potential: $\hat{\mathbf{B}}=\mathrm{grad}[Az(R^{-3}+2a^{-3})]$, and has a constant asymptotics $\hat{\mathbf{B}}\to 2a^{-3}A\hat{\mathbf{e}}_z$ as $R\to\infty$. The magnetic field lines are shown in Fig. 2 of Shafranov's paper [26].

The magnetic field lines (6.1) have two first integrals: the $\psi(r,z)$ and the angle ϕ. Hence any smooth function $f(\psi(r,z),\phi)$ also is their first integral. Inside the ball $R\leqslant a$, the magnetic field lines are either closed curves $\psi(r,z)=C_1$, $\phi=C_2$ or the separatrix $r=0$, $-a<z<a$ or the rest points: $r=a/\sqrt{2}$, $z=0$, $0\leqslant\phi\leqslant 2\pi$, where $\psi=-\ell$, $\ell=a^4c_0\mu/40$.

Let $a(\psi,\phi)>0$ and $\beta(\psi,\phi)$ be any smooth functions on the annulus $-\ell\leqslant\psi\leqslant 0$, $0\leqslant\phi\leqslant 2\pi$ such that $a(0,\phi)=1$ and $\beta(0,\phi)=0$. Applying the symmetry transforms (4.4) with functions $a_1(\mathbf{x})=a(\psi,\phi)$, $b_1(\mathbf{x})=\mathrm{ch}\beta(\psi,\phi)$, $c_1(\mathbf{x})=\mathrm{sh}\beta(\psi,\phi)$, where $\psi=\psi(r,z)$ (6.2) and $C_1=1$ to the Hill solution (6.1), we obtain new sub-Alfvenic MHD equilibria

$$\mathbf{B}_1=\mathrm{ch}\beta(\psi,\phi)\mathbf{B}, \quad \mathbf{V}_1=\frac{\mathrm{sh}\beta(\psi,\phi)}{a(\psi,\phi)\sqrt{\mu}}\mathbf{B}, \tag{6.4}$$

$$\rho_1(\mathbf{x})=a^2(\psi,\phi)\rho_0,$$

$$P_1(\mathbf{x})=p_0-c_0\psi-\mathrm{sh}^2\beta(\psi,\phi)\mathbf{B}^2/(2\mu)$$

inside the ball $R\leqslant a$. On the sphere $R=a$, the equilibria (6.4) coincide with Hill's solution (6.1), (6.2) and hence are continued in the outer space $R>a$ by the same Hill's plasma equilibrium (6.1), (6.3). It is evident that the equilibria (6.4) have toroidal magnetic surfaces $\psi(r,z)=\mathrm{const}$ and closed magnetic field lines and are nonsymmetric. Hence the symmetry transform (4.4) breaks the axial symmetry of Hill's solution.

VII. EXACT MHD EQUILIBRIA WITH CURRENT SHEETS

(I) The symmetry transforms (2.3) can be used also to construct exact MHD equilibria with current sheets. Suppose that the initial non-field-aligned equilibrium $\rho(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, $\mathbf{V}(\mathbf{x})$, $P(\mathbf{x})$ is smooth and that the functions $a(\mathbf{x})$, $b(\mathbf{x})$, $c(\mathbf{x})$ (2.3) are discontinuous at some magnetic surface S: $\psi(\mathbf{x})=\psi_0$. Let them be equal to $a_1(\mathbf{x})$, $b_1(\mathbf{x})$, $c_1(\mathbf{x})$ and $a_2(\mathbf{x})$, $b_2(\mathbf{x})$, $c_2(\mathbf{x})$ on the two sides of S. Applying the corresponding symmetry (2.3), we obtain a MHD equilibrium with a contact (or tangential) discontinuity on the surface S. Indeed, the resulting vector fields $\mathbf{B}_1$, $\mathbf{V}_1$ and $\mathbf{B}_2$, $\mathbf{V}_2$ (2.3) are tangent to the surface S on the both sides. As is known [27], the only necessary condition at the surface of contact discontinuity is

$$P_1+\mathbf{B}_1^2/(2\mu)=P_2+\mathbf{B}_2^2/(2\mu). \tag{7.1}$$

Using one of equations (2.3), $P_1=CP+(C\mathbf{B}^2-\mathbf{B}_1^2)/(2\mu)$, we obtain

$$P_1+\mathbf{B}_1^2/(2\mu)=C_1(P+\mathbf{B}^2/(2\mu)),$$

$$P_2+\mathbf{B}_2^2/(2\mu)=C_2(P+\mathbf{B}^2/(2\mu)),$$

where $C_j=b_j^2(\mathbf{x})-c_j^2(\mathbf{x})$, $j=1,2$. Hence the necessary condition (7.1) is satisfied if $C_1=C_2$. Thus we see that the symmetry transform (2.3) defines a new solution with a contact discontinuity at the magnetic surface $\psi(\mathbf{x})=\psi_0$ if $b_1^2(\mathbf{x})-c_1^2(\mathbf{x})=b_2^2(\mathbf{x})-c_2^2(\mathbf{x})$.

Any surface of contact discontinuity carries an electric current with the surface density [27] $\mathbf{J}(\mathbf{x})=\mu^{-1}\mathbf{n}(\mathbf{x})\times[\mathbf{B}_2(\mathbf{x})-\mathbf{B}_1(\mathbf{x})]$. Here $\mathbf{n}(\mathbf{x})$ is the unit normal vector field to the surface S directed from the side 1 to side 2. For the discontinuous symmetry transform (2.3), we have

$$\mathbf{J}(\mathbf{x})=\mu^{-1}\mathbf{n}(\mathbf{x})\times[(b_2-b_1)\mathbf{B}(\mathbf{x})+(c_2-c_1)\sqrt{\mu\rho}\mathbf{V}(\mathbf{x})], \tag{7.2}$$

where $\mathbf{B}(\mathbf{x})$, $\mathbf{V}(\mathbf{x})$, $\rho(\mathbf{x})$ characterize the initial MHD equilibrium. Hence for the noncollinear vector fields $\mathbf{V}$ and $\mathbf{B}$, we obtain

$$\begin{aligned}\mathbf{J}(\mathbf{x})=&Z^{-1}(\mathbf{x})[(b_2-b_1)\mathbf{B}\cdot\mathbf{B}+(c_2-c_1)\sqrt{\mu\rho}\mathbf{V}\cdot\mathbf{B}]\mathbf{V}\\&-Z^{-1}(\mathbf{x})[(b_2-b_1)\mathbf{V}\cdot\mathbf{B}+(c_2-c_1)\sqrt{\mu\rho}\mathbf{V}\cdot\mathbf{V}]\mathbf{B},\end{aligned} \tag{7.3}$$

where $Z(\mathbf{x})=\mu\sqrt{(\mathbf{V}\cdot\mathbf{V})(\mathbf{B}\cdot\mathbf{B})-(\mathbf{V}\cdot\mathbf{B})^2}$. Thus the magnetic surface S is a current sheet with the surface current (7.3).

For any function $F(\mathbf{x})$ having the limiting values $F_1(\mathbf{x})$ and $F_2(\mathbf{x})$ at the two sides of a surface S we denote the jump in F across S as $[F(\mathbf{x})]=F_2(\mathbf{x})-F_1(\mathbf{x})$. For the discontinuous transform (2.3), the plasma density and plasma velocity have the jumps

$$[\rho]=[a^2(\mathbf{x})]\rho,\quad [\mathbf{V}]=\left[\frac{c(\mathbf{x})}{a(\mathbf{x})}\right]\frac{1}{\sqrt{\mu\rho}}\mathbf{B}+\left[\frac{b(\mathbf{x})}{a(\mathbf{x})}\right]\mathbf{V} \tag{7.4}$$

at the surface of contact discontinuity S. For the stable steady solutions with a nonzero plasma viscosity and plasma diffusion, the jumps (7.4) have to be zero. This condition implies

$$a_2(\mathbf{x})=-a_1(\mathbf{x}),\quad b_2(\mathbf{x})=-b_1(\mathbf{x}),\quad c_2(\mathbf{x})=-c_1(\mathbf{x}). \tag{7.5}$$

For this case formulas (2.3) yield

$$\rho_2(\mathbf{x})=\rho_1(\mathbf{x}),\quad \mathbf{B}_2(\mathbf{x})=-\mathbf{B}_1(\mathbf{x}),$$
$$\mathbf{V}_2(\mathbf{x})=\mathbf{V}_1(\mathbf{x}),\quad P_2(\mathbf{x})=P_1(\mathbf{x}). \tag{7.6}$$

Hence only the magnetic field has a jump and the surface current $\mathbf{J(x)}$ (7.2) is

$$\mathbf{J}(\mathbf{x})=-2\mu^{-1}\mathbf{n}(\mathbf{x})\times[b_1(\mathbf{x})\mathbf{B}(\mathbf{x})+c_1(\mathbf{x})\sqrt{\mu\rho(\mathbf{x})}\mathbf{V}(\mathbf{x})], \tag{7.7}$$

(II) For the field-aligned solutions (4.1), the contact discontinuity can occur on any surface S that is invariant with respect to the magnetic dynamical system (4.2). Formulas (4.4) and the conditions $[\rho]=0$, $[\mathbf{V}]=0$ imply on the surface S,

$$a_{1.2}(\mathbf{x})=\pm a_{1.1}(\mathbf{x}),\quad b_{1.2}(\mathbf{x})=-b_{1.1}(\mathbf{x}),$$
$$c_{1.2}(\mathbf{x})=\pm c_{1.1}(\mathbf{x}). \tag{7.8}$$

The corresponding surface current (7.7) is

$$\mathbf{J}(\mathbf{x})=-2\mu^{-1}b_{1.1}(\mathbf{x})\mathbf{n}(\mathbf{x})\times\mathbf{B}(\mathbf{x}). \tag{7.9}$$

VIII. ASTROPHYSICAL JETS AS THE NONSYMMETRIC GLOBAL MHD EQUILIBRIA

(I) We start with the following helically symmetric [28] magnetic fields:

$$\mathbf{B}_h=\frac{\psi_u}{r}\hat{\mathbf{e}}_r+B_1\hat{\mathbf{e}}_z+B_2\hat{\mathbf{e}}_\phi,\quad B_1=\frac{\alpha\gamma\psi-r\psi_r}{r^2+\gamma^2},$$
$$B_2=\frac{\alpha r\psi+\gamma\psi_r}{r^2+\gamma^2}, \tag{8.1}$$

where $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_z$, $\hat{\mathbf{e}}_\phi$ are the unit orts in the cylindrical coordinates r, z, ϕ and $\psi=\psi(r,u)$ is the flux function, $u=z-\gamma\phi$, $\alpha=$const, $\gamma=$const. In Ref. [1], we obtained the exact plasma equilibria (8.1), $\mathrm{curl}\mathbf{B}\times\mathbf{B}=\mu\,\mathrm{grad}\,P$, $\mathrm{div}\,\mathbf{B}=0$, that correspond to the flux functions

$$\psi_{Nmn}=e^{-\beta r^2}\{a_N B_{0N}(y)+r^m B_{mn}(y)[a_{mn}\cos(mu/\gamma)$$
$$+b_{mn}\sin(mu/\gamma)]\}, \tag{8.2}$$

where N,m,n are arbitrary integers $\geqslant 0$ satisfying the inequality $2N>2n+m$, and $y=2\beta r^2$. The plasma pressure is $P=p_0-2\beta^2\psi^2/\mu$, and the plasma velocity $\mathbf{V}=0$. The polynomials $B_{mn}(y)$ have the form

$$B_{mn}(y)=\frac{\mathrm{d}^m}{\mathrm{d}y^m}L_{m+n}(y)-k_{mn}y\frac{\mathrm{d}^{m+1}}{\mathrm{d}y^{m+1}}L_{m+n}(y),$$

where $L_p(y)$ are the Laguerre polynomials

$$L_p(y)=\frac{1}{p!}e^y\frac{\mathrm{d}^p}{\mathrm{d}y^p}(y^pe^{-y})=\sum_{k=0}^{p}\frac{(-1)^kp!}{(k!)^2(p-k)!}y^k.$$

The simplest exact solution (8.2) is defined for $N=1$, $m=1$, $n=0$ and has the form

$$\psi_{110}(r,z,\phi)=e^{-\beta r^2}[1-4\beta r^2+a_1r\cos(z/\gamma-\phi)]. \tag{8.3}$$

(II) Applying the symmetry transforms (4.4) to the exact solutions (8.1) and (8.2) with $\mathbf{V}=0$, we obtain an infinite family of new field-aligned MHD equilibria for $C=k^2$,

$$\mathbf{B}_1=k\,\mathrm{ch}f(\mathbf{x})\mathbf{B}_h,\quad \mathbf{V}_1=\frac{k\,\mathrm{sh}f(\mathbf{x})}{\sqrt{\mu\rho_1(\mathbf{x})}}\mathbf{B}_h,$$
$$P_1=k^2P-\frac{k^2}{2\mu}\mathrm{sh}^2f(\mathbf{x})\mathbf{B}_h^2. \tag{8.4}$$

Here $f(\mathbf{x})$ and the plasma density $\rho_1(\mathbf{x})=a_1^2(\mathbf{x})$ are arbitrary smooth functions that are constant on the magnetic field lines (8.1) which all go to infinity in the variable z [1]. Therefore functions $f(\mathbf{x})$ and $a_1(\mathbf{x})$ depend on the two transversal variables and have no symmetry in general. Hence the generic exact solutions (8.4) are nonsymmetric.

(III) The nonsymmetric MHD equilibria (8.4) are sub-Alfvenic because the ratio of the plasma kinetic and magnetic energies is

$$\mu\rho_1\mathbf{V}_1^2/\mathbf{B}_1^2=\mathrm{th}^2f(\mathbf{x})<1. \tag{8.5}$$

This ratio is variable in the space $\mathbb{R}^3$ but is constant on the magnetic field lines.

The exact solutions (8.4) with discontinuous functions $f(\mathbf{x})$ have current sheets S that are invariant with respect to the magnetic field $\mathbf{B}_h$ lines. The generic invariant surface S has no geometrical symmetries. The necessary conditions (7.8) at the contact discontinuity surface S imply $f_2(\mathbf{x})=-f_1(\mathbf{x})$, $k_2=-k_1$. The surface S carries the electric current (7.9):

$$\mathbf{J}(\mathbf{x})=-2\mu^{-1}k_1\mathrm{ch}f_1(\mathbf{x})\mathbf{n}(\mathbf{x})\times\mathbf{B}_h(\mathbf{x}). \tag{8.6}$$

Using the known Galilean invariance [3,4] of the time-dependent MHD equations, we obtain from Eq. (8.4) the exact solutions

$$\mathbf{B}(\mathbf{x},t)=\mathbf{B}_1(x,y,z-v_0t),$$
$$\mathbf{V}(\mathbf{x},t)=\mathbf{V}_1(x,y,z-v_0t)+v_0\hat{\mathbf{e}}_z, \tag{8.7}$$
$$\rho(\mathbf{x},t)=\rho_1(x,y,z-v_0t),\quad P(\mathbf{x},t)=P_1(x,y,z-v_0t).$$

These solutions describe a nonsymmetric astrophysical jet that moves as a whole along the z axis with a constant speed v_0. Indeed, for the solutions (8.4), (8.7) with $f(\mathbf{x}) = c_\ell \psi^{2\ell+q}(\mathbf{x})$, $\rho_1(\mathbf{x}) = b_\ell \psi^{2\ell}(\mathbf{x})$, $\ell > 0$, $q \geqslant 0$, the plasma magnetic and kinetic energies and its mass are finite in any layer $c_1 \leqslant z \leqslant c_2$ because $\psi_{Nmn}(r,u) \approx c_N r^{2N} \exp(-\beta r^2)$ at $r \to \infty$. All magnetic field lines and plasma streamlines are bounded in the radial variable r because the leading term of the flux function $\psi_{Nmn}(r,u)$ (8.2) is $b_N(-2\beta r^2)^N \exp(-\beta r^2)/N!$ at $r \gg 1$, $b_N = a_N(1+k_{0N})$. In view of the rapid decreasing of the magnetic field $\mathbf{B}_h(r,u)$ as $r \to \infty$, the plasma magnetic and kinetic energies and its mass are concentrated near the z axis $r=0$. Hence we see that solutions (8.4), (8.7) are global and satisfy the necessary physical conditions (a), (b), (c) of Sec. I. Therefore the exact nonsymmetric solutions (8.4), (8.7) model the astrophysical jets outside of their accretion disks; for example, the jet in the elliptic galaxy Messier 87 [5–7]. These nonsymmetric exact solutions may have the current sheets with the surface current density (8.6).

Remark 7. On the stability of the MHD equilibria (8.4). The main method to study the nonlinear stability of the MHD equilibria is the energy variational method [2,29,30]. In the papers [24,25], it is shown that for the MHD equilibria with noncollinear magnetic field $\mathbf{B}$ and plasma velocity $\mathbf{V}$, the second variation of energy, $\delta^2\mathcal{H}$, is indefinite. The same is true for the aligned and super-Alfvenic flows. The only MHD equilibria whose stability can be proved by the energy variational method are the field-aligned and sub-Alfvenic flows. The solutions (8.4) are exactly of this type: they are field aligned and sub-Alfvenic, see formula (8.5).

Remark 8. On the preferable exact solutions. The class of exact MHD equilibria (8.4) is large: the real parameters a_N, a_{mn}, b_{mn}, $\beta > 0$, γ and the integer parameters N,n,m, $2N > 2n+m$ are arbitrary, the two first integrals $a_1(\mathbf{x})$ and $f(\mathbf{x})$ of the magnetic dynamical system (4.2), (8.1) are also arbitrary. Hence a natural question arises: what values of these parameters are preferable for the model of the real astrophysical jets? The available observational data [5–7] are not sufficient to answer this question completely. However, they imply that the most preferable is the ground state solution (8.3) with only two magnetic axes [1]. Thus the integer parameters are $N=1$, $m=1$, $n=0$. The preferable asymptotics of the functions $a_1(\mathbf{x})$ and $f(\mathbf{x})$ as $r \to \infty$ are

$$a_1(\mathbf{x}) \to 0, \quad f(\mathbf{x}) \to 0, \quad f(\mathbf{x})/a_1(\mathbf{x}) \to 0,$$

$$\int \int_{R^2} a_1^2(x,y)\,\mathrm{d}x\,\mathrm{d}y < C_0 .$$

For this case, the ratio (8.5) of the plasma kinetic and magnetic energies th^2 $f(\mathbf{x}) \to 0$, $|\mathbf{V}_1| \ll |\mathbf{B}_h|/\mu$, $\rho_1(\mathbf{x}) \to 0$ as $r \to \infty$, and the total mass of plasma and its kinetic and magnetic energies are finite in any layer $c_1 < z < c_2$.

IX. MHD EQUILIBRIA WITH NONCOLLINEAR VECTOR FIELDS B AND V

(I) The known Grad's ''transverse'' flows [31] are cylindrically symmetric and describe the differential rotation of a perfectly conducting ideal gas plasma around the axis z in the vertical magnetic field,

$$\mathbf{B} = H(r)\hat{\mathbf{e}}_z, \quad \mathbf{V} = \omega(r)(-y\hat{\mathbf{e}}_x + x\hat{\mathbf{e}}_y),$$
$$P(r) = F(r) - H^2(r)/(2\mu). \tag{9.1}$$

Here $F(r) = \int_0^r t\rho(t)\omega^2(t)dt + \varepsilon^2$. The solutions (9.1) depend on the three arbitrary functions $\omega(r)$, $H(r)$, $\rho(r) \geqslant 0$ and satisfy the ideal gas equation of state (1.3). The density of entropy $S(r)$ is defined from the Eqs. (1.3) and (9.1) for the arbitrary gas density $\rho(r)$ and the adiabatic exponent $\gamma > 1$.

Applying the symmetry transforms (2.3) to the solutions (9.1), we obtain the new exact MHD equilibria

$$\rho_1(r) = a^2(r)\rho(r), \quad \mathbf{B}_1 = c\sqrt{\mu\rho}\mathbf{V} + bH\hat{\mathbf{e}}_z,$$

$$\mathbf{V}_1 = \frac{b}{a}\mathbf{V} + \frac{c}{a\sqrt{\mu\rho}}H\hat{\mathbf{e}}_z, \tag{9.2}$$

where $b^2(r) - c^2(r) = C = \text{const}$ and $P_1 = CF - b^2H^2/(2\mu) - \rho(cr\omega)^2/2$. For the equilibria (9.2), the vector fields $\mathbf{B}_1$ and $\mathbf{V}_1$ are noncollinear. Exact solutions (9.2) satisfy the ideal gas equation of state of the form (1.3) and depend on four arbitrary functions $a(r)$, $b(r)$, $\omega(r)$, and $H(r)$. Their magnetic field lines and plasma streamlines are helices that lie on the cylindrical magnetic surfaces $x^2+y^2=r^2=\text{const}$.

We consider solutions (9.2) inside a cylinder $0 \leqslant r \leqslant r_1$ provided that $c(r_1)=0$, $\omega(r_1)=0$, $H(r_1)=0$ and the inequality $H^2(r) < 2\mu F(r)$ holds, then $P(r) > 0$. At the boundary $r=r_1$, we have $\mathbf{V}_1(r_1)=\mathbf{0}$, $\mathbf{B}_1(r_1)=\mathbf{0}$, $\rho_1(r_1)=\rho_0$, $P_1(r_1)=F(r_1)=p_0$; therefore the solution is continued in the outer space by the trivial solution $\mathbf{V}=\mathbf{0}$, $\mathbf{B}=\mathbf{0}$, $\rho=\rho_0$, $P=p_0$. Hence the solutions (9.2) describe the helical dynamics of plasma inside the cylinder.

Suppose that the functions $a(r)$, $b(r)$, $c(r)$ have a jump at $r=r_0$ that satisfies the necessary conditions (7.5). Then solution (9.2) has a contact discontinuity (7.6) at the cylinder $r=r_0$. This surface is a current sheet with the current $\mathbf{J}$ defined by the equations (7.7) and (9.1): $\mu\mathbf{J}(x,y,z) = -2b_1(r_0)H(r_0)(y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y) - 2c_1(r_0)\sqrt{\mu\rho(r_0)}\omega(r_0)\hat{\mathbf{e}}_z$. Hence we see that the current $\mathbf{J}$ lines are helices. The solutions (9.2) can have an arbitrary number of such cylindrical surfaces $r=r_k$ of contact discontinuities.

(II) As is shown in Remark 1 of Sec. II, the symmetry transforms (2.3) convert a given MHD equilibrium into another equilibria with the same magnetic surfaces. Hence the following question arises: *if* u, u_1 *are two* MHD *equilibria with the same magnetic surfaces, is there a symmetry transform* (2.3) *bringing them into each other*? We prove that, in general, such transform does not exist. Indeed, let us consider another Grad's ''transverse'' flow (9.1),

$$\mathbf{B}_2 = H_2(r)\hat{\mathbf{e}}_z, \quad \mathbf{V}_2 = \omega_2(r)(-y\hat{\mathbf{e}}_x + x\hat{\mathbf{e}}_y),$$
$$P_2(r) = F_2(r) - H_2^2(r)/(2\mu), \tag{9.3}$$

with $H_2(r)/H(r)\neq\text{const}$ and $\rho_2(r)>0$, $F_2(r)=\int_0^r t\rho_2(t)\omega_2^2(t)dt$. The two MHD equilibria (9.1) and (9.3) have the same cylindrical magnetic surfaces $r=\text{const}$. By applying to Eq. (9.1) the symmetry transforms (2.3) with $b^2(r)-c^2(r)=C=\text{const}$ we obtain the equilibria (9.2). If $c(r)\neq 0$ then all plasma streamlines are helices. If $c(r)=0$ then $b(r)=\sqrt{C}=\text{const}$, $\mathbf{B}_1=\sqrt{C}H(r)\hat{\mathbf{e}}_z$, and therefore $H_1(r)/H(r)=\sqrt{C}=\text{const}$. Hence it follows that the equilibrium (9.1) cannot be transformed by the symmetries (2.3) into the equilibrium (9.3) because for the latter all plasma streamlines are closed curves $z=\text{const}$, $r=\text{const}$, and $H_2(r)/H(r)\neq\text{const}$.

As a corollary of the group structure of G_m (see Sec. III), we find that no MHD equilibrium obtained from (9.3) by the symmetries (2.3) can be transformed by these symmetries into the equilibrium (9.2). However, all these equilibria have the same cylindrical magnetic surfaces $r=\text{const}$. Hence we conclude that, in general, the action of the groups of symmetries G_m (3.3) on the sets E_m is not transitive (E_m is the set of the MHD equilibria with the same magnetic surfaces).

(III) Let us show that any axially symmetric plasma equilibrium (5.1): $\mathbf{B}_a=[\psi_z\hat{\mathbf{e}}_r-\psi_r\hat{\mathbf{e}}_z+I(\psi)\hat{\mathbf{e}}_\phi]/r$, $\mathbf{V}=0$, $P=P_0(\psi)$ generates a family of MHD equilibria with noncollinear vector fields $\mathbf{B}$ and $\mathbf{V}$. Let $f(\psi)>0$ be a smooth function of ψ. It is easy to verify that functions ($\rho_0=\text{const}$)

$$\mathbf{B}=\mathbf{B}_a,\quad \mathbf{V}=rf(\psi)\hat{\mathbf{e}}_\phi,\quad P=P_0(\psi)+\rho_0 r^2,$$

$$\rho=2\rho_0 f^{-2}(\psi) \tag{9.4}$$

define a new MHD equilibrium. Indeed, formulas (9.4) imply $\rho(\mathbf{V}\cdot\text{grad})\mathbf{V}=-\rho_0\,\text{grad}\,r^2$, hence Eq. (1.1) follows. The Eqs. (1.2) are true because $\text{div}\,\mathbf{V}=\text{div}(\rho\mathbf{V})=0$ and $\mathbf{V}\times\mathbf{B}=-f(\psi)(\psi_r\hat{\mathbf{e}}_r+\psi_z\hat{\mathbf{e}}_z)$, hence $\text{curl}(\mathbf{V}\times\mathbf{B})=\mathbf{0}$. For the equilibrium (9.4), the magnetic surfaces are given by the equation $\psi(r,z)=\text{const}$. Applying the symmetry transforms (2.3) to the equilibrium (9.4) we obtain a continuous family of the axially symmetric MHD equilibria

$$\mathbf{B}_1=b(\mathbf{x})\mathbf{B}_a+mc(\mathbf{x})r\hat{\mathbf{e}}_\phi,$$

$$\mathbf{V}_1=\frac{f(\psi)}{a(\mathbf{x})}[m^{-1}c(\mathbf{x})\mathbf{B}_a+b(\mathbf{x})r\hat{\mathbf{e}}_\phi],$$

where $m=\sqrt{2\mu\rho_0}$, $\rho_1(\mathbf{x})=2\rho_0 a^2(\mathbf{x})/f^2(\psi)$, $b^2(\mathbf{x})-c^2(\mathbf{x})=C$, and $P_1=C(P_0(\psi)+\rho_0 r^2)+(C\mathbf{B}_a^2-\mathbf{B}_1^2)/(2\mu)$. The vector fields $\mathbf{B}_1$ and $\mathbf{V}_1$ are noncollinear.

(IV) Any helically symmetric plasma equilibrium $\mathbf{B}_h$ (8.1), $\mathbf{V}=0$, $P=P_0(\psi)$ also generates a family of the MHD equilibria with noncollinear vector fields $\mathbf{B}$ and $\mathbf{V}$. Indeed, we let

$$\mathbf{B}=\mathbf{B}_h,\quad \mathbf{V}=f(\psi)\mathbf{H},\quad P=P_0(\psi)+\rho_0 r^2,$$

$$\rho=2\rho_0 f^{-2}(\psi), \tag{9.5}$$

where $\mathbf{H}=\gamma\hat{\mathbf{e}}_z+r\hat{\mathbf{e}}_\phi$, $\psi=\psi(r,u)$ is the flux function for the equilibrium (8.1), $u=z-\gamma\phi$ and $f(\psi)>0$ is an arbitrary smooth function. A calculation shows that $\rho(\mathbf{V}\cdot\text{grad})\mathbf{V}=-\rho_0\,\text{grad}\,r^2$, hence Eq. (1.1) holds. Equations (1.2) are satisfied because $\text{div}\,\mathbf{V}=\text{div}(\rho\mathbf{V})=0$ and $\mathbf{V}\times\mathbf{B}=-f(\psi)\times(\psi_r\hat{\mathbf{e}}_r+\psi_u\hat{\mathbf{e}}_z-\gamma\psi_u\hat{\mathbf{e}}_\phi/r)=-f(\psi)\text{grad}\,\psi$, hence $\text{curl}(\mathbf{V}\times\mathbf{B})=\mathbf{0}$. The equation $\psi(r,u)=\text{const}$ defines the magnetic surfaces for the equilibrium (9.5). By applying the symmetry transforms (2.3) to the equilibrium (9.5) we get a family of the helically symmetric MHD equilibria,

$$\mathbf{B}_1=b(\mathbf{x})\mathbf{B}_h+mc(\mathbf{x})\mathbf{H},\quad \mathbf{V}_1=\frac{f(\psi)}{a(\mathbf{x})}[m^{-1}c(\mathbf{x})\mathbf{B}_h+b(\mathbf{x})\mathbf{H}]$$

with the noncollinear vector fields $\mathbf{B}_1$ and $\mathbf{V}_1$. Here $m=\sqrt{2\mu\rho_0}$, $\rho_1(\mathbf{x})=2\rho_0 a^2(\mathbf{x})/f^2(\psi)$, and $P_1=C(P_0(\psi)+\rho_0 r^2)+(C\mathbf{B}_h^2-\mathbf{B}_1^2)/(2\mu)$.

X. THE SYMMETRY TRANSFORMS FOR THE COMPRESSIBLE GAS PLASMA

For the ideal compressible gas plasma, Eqs. (1.1)–(1.3) are invariant under the following symmetry transforms:

$$\rho_1(\mathbf{x})=a^2(\mathbf{x})\rho(\mathbf{x}),\quad \mathbf{B}_1=b\mathbf{B},\quad \mathbf{V}_1=\frac{b}{a(\mathbf{x})}\mathbf{V}, \tag{10.1}$$

$$P_1=b^2P,\quad S_1=S+2C_V[\ln|b|-\gamma\ln|a(\mathbf{x})|],$$

where $a(\mathbf{x})\neq 0$ is an arbitrary smooth function that is constant on the magnetic field lines and on the plasma streamlines and $b=\text{const}\neq 0$. Transforms (10.1) preserve the equation of state (1.3).

Suppose that for a smooth non-field-aligned MHD equilibrium with an equation of state $P=\rho^\gamma f(S)$ the magnetic surfaces are closed in some domain D. Then for the generic case the entropy density $S(\mathbf{x})$ is constant on the magnetic surfaces and there exists a symmetry (10.1) that transforms the MHD equilibrium in the domain D into an isoentropic equilibrium satisfying the equation $P_1=b^2\rho_1^\gamma$.

Function $a(\mathbf{x})$ satisfies the equations

$$\mathbf{B}\cdot\text{grad}\,a(\mathbf{x})=0,\quad \mathbf{V}\cdot\text{grad}\,a(\mathbf{x})=0. \tag{10.2}$$

The latter equation implies

$$\rho_1(\mathbf{V}_1\cdot\text{grad})\mathbf{V}_1=b^2\rho(\mathbf{V}\cdot\text{grad})\mathbf{V},$$

$$\text{div}(\rho_1\mathbf{V}_1)=a(\mathbf{x})b\,\text{div}(\rho\mathbf{V})=0.$$

Hence Eqs. (1.1), (1.3) and the first equation (1.2) are satisfied. The only nontrivial equation is the third equation of Eq. (1.2). We have

$$\text{curl}(\mathbf{V}_1\times\mathbf{B}_1)=\text{curl}\left(\frac{b^2}{a(\mathbf{x})}\mathbf{V}\times\mathbf{B}\right)=\text{grad}\frac{b^2}{a(\mathbf{x})}\times(\mathbf{V}\times\mathbf{B})+\frac{b^2}{a(\mathbf{x})}\text{curl}(\mathbf{V}\times\mathbf{B}).$$

The first two equations (10.2) imply $\mathrm{grad}(b^2/a(\mathbf{x}))\times(\mathbf{V}\times\mathbf{B})=0$. Hence the equation $\mathrm{curl}(\mathbf{V}_1\times\mathbf{B}_1)=\mathbf{0}$ follows.

If for the MHD equilibrium ρ, $\mathbf{B}$, $\mathbf{V}$, P the magnetic surfaces are closed in some domain D then they are tori $\mathbb{T}^2$ [2]. In the Remark 9 below, we prove that the vector fields $\rho^{-1}\mathbf{B}$ and $\mathbf{V}$ commute. Hence the magnetic field lines and the plasma streamlines are dense quasiperiodic trajectories on the generic tori $\mathbb{T}^2$ [2]. Therefore any function that is constant along the plasma streamlines is constant on the tori $\mathbb{T}^2$. Hence equation $[\mathbf{V}\cdot \mathrm{grad}\, S(\mathbf{x})]=0$ (1.3) implies that entropy $S(\mathbf{x})$ is constant on the magnetic surfaces.

Suppose that plasma satisfies an equation $P=\rho^\gamma f(S)$. The function $a(\mathbf{x})=f^{1/(2\gamma)}(S(\mathbf{x}))$ is constant on the magnetic surfaces $\mathbb{T}^2$. The corresponding symmetry (10.1) transforms the equation of state $P=\rho^\gamma f(S)$ into the isoentropic equation $P_1=b^2\rho_1^\gamma$.

The symmetry transforms (10.1) form the subgroup $G_{0m}=A_m\oplus R^+\oplus Z_2\oplus Z_2\subset G_m$, see Eq. (3.3). Elements of G_{0m} are the sextuples $[\alpha(\mathbf{x}),0,k,\tau,1,\eta]$. The subgroup G_{0m} has the additional structure of a module over the associative algebra A_m with multiplication induced by Eq. (3.4).

Remark 9. Let us prove that for any variable plasma density $\rho(\mathbf{x})$ vector fields $\rho^{-1}\mathbf{B}$ and $\mathbf{V}$ commute, or their commutator $[\rho^{-1}\mathbf{B},\mathbf{V}]=0$. Indeed, the known identity $\mathrm{curl}(\mathbf{X}\times\mathbf{Y})=(\mathrm{div}\,\mathbf{Y})\mathbf{X}-(\mathrm{div}\,\mathbf{X})\mathbf{Y}+[\mathbf{Y},\mathbf{X}]$ and the equation $\mathrm{curl}(\mathbf{V}\times\mathbf{B})=\mathbf{0}$ imply

$$\mathrm{curl}(\rho\mathbf{V}\times\rho^{-1}\mathbf{B})=\mathrm{div}(\rho^{-1}\mathbf{B})\rho\mathbf{V}-\mathrm{div}(\rho\mathbf{V})\rho^{-1}\mathbf{B}+[\rho^{-1}\mathbf{B},\rho\mathbf{V}]=0.$$

Substituting equation $\mathrm{div}(\rho\mathbf{V})=0$ we get

$$[\rho^{-1}\mathbf{B},\rho\mathbf{V}]+\rho\,\mathrm{div}(\rho^{1-}\mathbf{B})\mathbf{V}=0. \tag{10.3}$$

Equation $\mathrm{div}\,\mathbf{B}=0$ implies $\mathrm{div}(\rho\rho^{-1}\mathbf{B})=\rho\,\mathrm{div}(\rho^{-1}\mathbf{B})+\rho^{-1}(\mathbf{B}\cdot\mathrm{grad}\,\rho)=0$. This equation yields

$$\rho[\rho^{-1}\mathbf{B},\mathbf{V}]=[\rho^{-1}\mathbf{B},\rho\mathbf{V}]-\rho^{-1}(\mathbf{B}\cdot\mathrm{grad}\,\rho)\mathbf{V}=[\rho^{-1}\mathbf{B},\rho\mathbf{V}]+\rho\,\mathrm{div}(\rho^{-1}\mathbf{B})\mathbf{V}.$$

Hence, using Eq. (10.3) we obtain $[\rho^{-1}\mathbf{B},\mathbf{V}]=0$.

XI. INVARIANTS OF THE SYMMETRY TRANSFORMS

(I) The symmetry transforms (2.3) [and (10.1)] have the following physical meaning. The difference between the plasma kinetic and magnetic energies is changed by a scalar multiplication. Indeed, the transform (2.3) implies

$$\frac{1}{2}\rho_1\mathbf{V}_1^2-\frac{1}{2\mu}\mathbf{B}_1^2=C\left(\frac{1}{2}\rho\mathbf{V}^2-\frac{1}{2\mu}\mathbf{B}^2\right), \tag{11.1}$$

where $C=b^2(\mathbf{x})-c^2(\mathbf{x})=\mathrm{const}$ [$C=b^2$ for the symmetries (10.1)]. Another consequence of transform (2.3) is the equation $\mathbf{V}_1\times\mathbf{B}_1=C\mathbf{V}\times\mathbf{B}/a(\mathbf{x})$. Hence we obtain $\sqrt{\rho_1}\mathbf{V}_1\times\mathbf{B}_1=C\sqrt{\rho}\mathbf{V}\times\mathbf{B}$, or $\sqrt{\rho_1}\mathbf{E}_1=C\sqrt{\rho}\mathbf{E}$. Here $\mathbf{E}=-\mathbf{V}\times\mathbf{B}/c_0$ is the electric field for the plasma with a perfect electric conductivity; c_0 is the speed of light. Hence the symmetries (2.3) have the vector field invariant

$$\frac{\sqrt{\rho_1}\mathbf{V}_1\times\mathbf{B}_1}{\rho_1\mathbf{V}_1^2-\mathbf{B}_1^2/\mu}=\frac{\sqrt{\rho}\mathbf{V}\times\mathbf{B}}{\rho\mathbf{V}^2-\mathbf{B}^2/\mu}.$$

This invariant implies that the symmetry transforms (2.3) preserve magnetic surfaces and the integrable foliation (2.2): $d\psi(\mathbf{Y}(\mathbf{x}))=0$.

Remark 10. An application of the Newcomb variational principle. In his 1962 paper [32], Newcomb proved that the (time-dependent) MHD equations (1.1) follow from the variational principle,

$$\delta\int_{t_1}^{t_2}\mathrm{d}t\int L(\mathbf{B},\mathbf{V},\rho)\mathrm{d}^3x=0,\quad L(\mathbf{B},\mathbf{V},\rho)=\frac{1}{2}\rho\mathbf{V}^2-\frac{1}{2\mu}\mathbf{B}^2, \tag{11.2}$$

provided that the (time-dependent) equations (1.2) are satisfied. The symmetry transforms (2.3) preserve the equations (1.2) because the functions $a(\mathbf{x})$, $b(\mathbf{x})$, and $c(\mathbf{x})$ are constant on the magnetic field lines and on the plasma streamlines. Equation (11.1) implies the following relation between the Lagrangians:

$$L(\mathbf{B}_1,\mathbf{V}_1,\rho_1)=CL(\mathbf{B},\mathbf{V},\rho). \tag{11.3}$$

Equation (11.3) means that the symmetry transforms (2.3) preserve the Lagrangian of the Newcomb variational principle (11.2) up to a constant factor. Hence any extremum of the principle (11.2) is transformed into a new extremum. Thus we obtain the second proof of the fact that symmetries (2.3) [and (10.1)] transform any solution of equations (1.1) and (1.2) into new solutions. The first proof of Sec. II is straightforward and independent of Newcomb's variational principle.

XII. SUMMARY

The method for constructing exact magnetohydrodynamics equilibria consists of the application of symmetry transforms (2.3) and (10.1) to any known equilibrium. The method is applicable to the divergence-free flows of plasma and to the ideal gas plasma flows with $\mathrm{div}\,\mathbf{V}\neq 0$. The symmetry transforms have form (2.3) for $\mathrm{div}\,\mathbf{V}=0$ and depend upon two arbitrary functions $a(\mathbf{x})$ and $b(\mathbf{x})$ that are constant on the magnetic field lines and on the plasma streamlines. For the ideal gas plasma flows with $\mathrm{div}\,\mathbf{V}\neq 0$, the symmetries have a simpler form (10.1) and depend upon one arbitrary function $a(\mathbf{x})$ that is constant on the magnetic field lines and on the plasma streamlines. For any physical solution to equations (1.1) and (1.2), the method of symmetry transforms (2.3) and (10.1) gives a continuous family of new solutions.

The symmetry transforms (2.3) and (10.1) break the geometrical symmetries of the field-aligned MHD equilibria in the domains D where all magnetic field lines either are closed curves or go to infinity. By applying the symmetry transforms (2.3), (4.4) to the magnetic analog of Hill's spherical vortex [14] we have obtained a continuous family of nonsymmetric MHD equilibria with toroidal magnetic surfaces and closed magnetic field lines. We have derived the exact MHD equilibria with current sheets by applying to the smooth MHD equilibria the symmetries (2.3) with discontinuous functions $a(\mathbf{x})$, $b(\mathbf{x})$, $c(\mathbf{x})$ that satisfy the necessary conditions (7.5).

By applying the method of symmetry transforms (2.3), we have derived the exact solutions (5.3) that model ball lightning with dynamics of plasma inside the fireball and a large family (8.4) of global well-behaved nonsymmetric MHD equilibria. These exact solutions satisfy all necessary physical conditions (a), (b), (c) of Sec. I and model the astrophysical jets outside of their accretion disks, for example, the jet in the elliptic galaxy Messier 87 [5–7]. The total plasma kinetic and magnetic energy and its mass are finite in any layer $c_1 < z < c_2$. The exact solutions may have the current sheets with the surface current density (8.6).

The method of symmetry transforms (2.3) has the following features that distinguish it from the method of Backlund transforms for the soliton equations:

(i) The method of symmetry transforms gives new solutions in explicit algebraic form.

(ii) The symmetry transforms (2.3) depend on all three spatial variables $\mathbf{x} = x, y, z$.

(iii) The generic transforms (2.3) break the geometrical symmetries of the field-aligned equilibria.

(iv) The symmetries (2.3) form infinite-dimensional Abelian groups G_m (3.3) that depend on the topology of the MHD equilibria.

(v) The groups of symmetries G_m (3.3) have the additional structure (3.4) of modules over the associative algebras of functions.

ACKNOWLEDGMENTS

The author thanks Benno Fuchssteiner, Richard Henriksen, and Harold Weitzner for useful discussions.

[1] O. I. Bogoyavlenskij, Phys. Rev. E **62**, 8616 (2000).
[2] M. D. Kruskal and R. M. Kulsrud, Phys. Fluids **1**, 265 (1958).
[3] D. Biskamp, *Nonlinear Magnetohydrodynamics* (Cambridge University Press, Cambridge, 1993).
[4] J. F. Freidberg, *Ideal Magnetohydrodynamics* (Plenum, New York, 1987).
[5] D. C. Hines, F. N. Owen, and J. A. Eilek, Astrophys. J. **347**, 731 (1989).
[6] T. P. Ray, *Astrophysical Jets. Open Problems* (Gordon and Breach, Amsterdam, 1998), p. 173.
[7] A. Ferrari, Annu. Rev. Astron. Astrophys. **36**, 539 (1998).
[8] M. L. Woolley, J. Plasma Phys. **17**, 259 (1977).
[9] B. K. Shivamoggi, Q. Appl. Math. **XLIV**, 487 (1986).
[10] A. Salat and R. Kaiser, Phys. Plasmas **2**, 3777 (1995).
[11] R. Kaiser and A. Salat, Phys. Rev. Lett. **77**, 3133 (1996).
[12] H. Wu and Y. Chen, Phys. Fluids B **1**, 1753 (1989).
[13] R. Kaiser and D. Lortz, Phys. Rev. E **52**, 3034 (1995).
[14] J. Hill, Philos. Trans. R. Soc. London, Ser. A **185**, 213 (1894).
[15] O. I. Bogoyavlenskij, Phys. Rev. Lett. **84**, 1914 (2000).
[16] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. **19**, 1095 (1967).
[17] B. B. Kadomtsev and V. P. Petviashvili, Sov. Phys. Dokl. **15**, 539 (1970).
[18] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. **31**, 125 (1973).
[19] H. K. Moffatt, *Turbulence and Nonlinear Dynamics in MHD Flows* (North-Holland, Amsterdam, 1989), p. 185.
[20] S. Chandrasekhar, Proc. Natl. Acad. Sci. U.S.A. **42**, 273 (1956).
[21] U. Gebhardt and M. Kiessling, Phys. Fluids B **4**, 1689 (1992).
[22] P. Spiess and K. Elsasser, Phys. Plasmas **6**, 4208 (1999).
[23] O. I. Bogoyavlenskij, Phys. Lett. A **276**, 257 (2000).
[24] E. Hameiri, Phys. Plasmas **5**, 3270 (1998).
[25] S. Friedlander and M. M. Vishik, Chaos **5**, 416 (1995).
[26] V. D. Shafranov, JETP **6**, 545 (1958).
[27] H. Weitzner, in *Handbook of Plasma Physics*, edited by M. N. Rosenbluth and R. Z. Sagdeev (North-Holland, Amsterdam, 1983), p. 201.
[28] J. L. Johnson, C. R. Oberman, R. M. Kulsrud, and E. A. Frieman, Phys. Fluids **1**, 281 (1958).
[29] L. Woltjer, Proc. Natl. Acad. Sci. U.S.A. **44**, 833 (1958).
[30] V. I. Arnold, Appl. Math. Mech. **29**, 1002 (1965).
[31] H. Grad, Rev. Mod. Phys. **32**, 830 (1960).
[32] W. A. Newcomb, Nucl. Fusion Suppl. **2**, 451 (1962).